Semilattices, Domains, and Computability

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Algebraic Lattices:
The Prime Examples

- The ***powerset*** of a set.
- The lattice of ***subgroups*** of a group.
- The lattice of ***ideals*** of a ring.

What is a Lattice?

$0 \leq x \leq 1$ — **Bounded**

$x \leq x$

$x \leq y \;\&\; y \leq z \Rightarrow x \leq z$

$x \leq y \;\&\; y \leq x \Rightarrow x = y$

Partially Ordered Set

$x \vee y \leq z \Leftrightarrow x \leq z \;\&\; y \leq z$

$z \leq x \wedge y \Leftrightarrow z \leq x \;\&\; z \leq y$

With sups & With infs

What is a Semilattice?

$0 \leq x \leq 1$	**Bounded**
$x \leq x$	**Partially**
$x \leq y \,\&\, y \leq z \Rightarrow x \leq z$	**Ordered**
$x \leq y \,\&\, y \leq x \Rightarrow x = y$	**Set**
$x \vee y \leq z \Leftrightarrow x \leq z \,\&\, y \leq z$	**With sups**

What is a Complete Semiattice?

$$\bigvee_{i\in I} x_i \leq y \Leftrightarrow (\forall i \in I)\ x_i \leq y$$

What is a Complete Lattice?

$$y \leq \bigwedge_{i\in I} x_i \Leftrightarrow (\forall i \in I)\ y \leq x_i$$

Note: A complete semilattice is a complete lattice:

$$\bigwedge_{i\in I} x_i = \bigvee \{y \mid (\forall i \in I)\ y \leq x_i\}$$

The Equational Axiomatization of Semilattices

$$0 \vee x = x \qquad 1 \vee x = 1 \qquad x \vee x = x$$

$$x \vee y = y \vee x$$

$$x \vee (y \vee z) = (x \vee y) \vee z$$

Definition: $x \leq y \Leftrightarrow x \vee y = y$

Algebraic Lattices:
The Abstract Definition

Definition: An element u of a complete lattice is ***finite*** (or, ***compact***) provided that whenever we have $u \leq \bigvee_{i\in I} x_i$, then $u \leq \bigvee_{i\in J} x_i$ for some ***finite*** $J \subseteq I$.

Definition: A complete lattice is ***algebraic*** iff every element is the sup of its finite subelements.

Note: **The finite elements of the lattice of subgroups of a group are exactly the finitely generated subgroups. And the lattice is thus algebraic.**

Semilattice Completion

Theorem: The finite elements of a complete lattice form a subsemilattice — provided the unit element is finite.

Definition: The ***ideals*** of semilattice are the subsets closed under finite sups and subelements.

Theorem: The ideals of semilattice form an algebraic lattice with a finite unit element.

Theorem: Every algebraic lattice with a finite unit element is isomorphic to the ideal lattice of its semilattice of finite elements.

Topological Connections

Theorem: Every algebraic lattice becomes a ***T_0-topological space*** with a basis for the open sets consisting of sets $\uparrow u = \{ x \mid u \leq x \}$ for u finite.

Theorem: The lattice of open subsets of the ***Cantor Discontinuum*** is an algebraic lattice with the finite elements being the compact opens.

Theorem: The ***continuous functions*** between algebraic lattices are exactly the functions preserving ***directed sups***. They can also be characterized by the equation:

$$F(x) = \bigvee \{ F(u) \mid u \leq x \ \&\ u \text{ finite} \}.$$

What are Scott-Ershov Domains?

Definition: A ***domain*** is an algebraic lattice ***minus*** a finite unit; ***equivalently ...***

A ***domain*** is any ***closed subset*** of an algebraic lattice; ***equivalently ...***

A ***domain*** is the completion of a semilattice by ***proper ideals***.

Note: Every algebraic lattice is a domain.

(Hint: Add an extra unit element at the top.)

Theorem: Domains form a ***category*** with the continuous functions as the ***mappings***.

Back to Semilattices!

Definition: For $\mathcal{A} = \langle A, 0, 1, \vee \rangle$ a given semilattice,

let $||\mathcal{A}||$ be the set of ***proper ideals*** of $\mathcal{A}$; that is,

$||\mathcal{A}|| = \{X \subseteq A \mid 0 \in X \,\&\, 1 \notin X \,\&\, \forall a,b \in A[a,b \in X \Leftrightarrow a \vee b \in X]\}$.

Theorem: $||\mathcal{A}||$ is a domain with finite elements of the form

$\downarrow a = \{x \in A \mid a \vee x = a\}$; if additionally, $\mathcal{A}$ satisfies

$$\forall a,b \in A[a = 1 \text{ or } b = 1 \Leftrightarrow a \vee b = 1],$$

then $||\mathcal{A}||$ is an algebraic lattice.

Note: Intuitively 0 indicates no information and 1 too much information or an inconsistency.

The Countable Case

Theorem: The completion $||\mathcal{A}||$ of a ***countable*** semilattice $\mathcal{A} = \langle A, 0, 1, \vee \rangle$ can be thought of as adding ***limit points*** to $\mathcal{A}$ of increasing sequences $a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$ of elements of A, where we define $\mathbf{lim}_{n\to\infty}\, a_n \leq \mathbf{lim}_{m\to\infty}\, b_m$ to mean that ***each*** a_n ***is*** $\leq$ ***some*** b_m.

Note: **Of course, limits prove to be sups in $||\mathcal{A}||$, and we can identify the elements of A with the limits of the constant sequences. However, from this point of view, in order to prove that $||\mathcal{A}||$ is (directed) complete, it is probably easier to relate limits to ideals.**

A Universal Semilattice

Definition: Let $\mathcal{P} = \langle P, 0, 1, \vee \rangle$ be the ***semilattice*** of (equivalence classes of) ***propositional formulae*** with generators $\xi_0, \xi_1, \xi_2, \ldots, \xi_n, \ldots$.

Note: **We will use the usual notation for other propositional operators, so $\mathcal{P}$ may also be considered a Boolean algebra.**

Theorem: $\|\mathcal{P}\|$ as a domain is isomorphic to the domain of ***proper open subsets*** of the Cantor set.

Main Theorem: ***Every*** domain with a countable number of finite elements can be ***isomorphically embedded*** into $\|\mathcal{P}\|$.

Outline of a Proof

Theorem: Every ***countable Boolean algebra*** can be isomorphically embedded into $\mathcal{P}$.

Hint: It is easy to show that a finite Boolean algebra can be embedded into $\mathcal{P}$. And then the embedding can be continued to any finite superalgebra. Next note that a countable algebra is the union of a countable chain of finite algebras.

Theorem: Every ***countable semilatice*** can be isomorphically embedded into $\mathcal{P}$.

Hint: Every semilattice $\mathcal{A} = \langle A, 0, 1, \vee \rangle$ can be embedded into the powerset lattice of $A \backslash \{1\}$ by the mapping $\rho(a) = \{x \mid a \not\leq x\}$.

Theorem: If a semilattice $\mathcal{A}$ is a subsemilattice of a semilattice $\mathcal{B}$, then $\|\mathcal{A}\|$ is a ***subdomain*** of $\|\mathcal{B}\|$.

Simplifying the Notation

(Step 1) $<P, 0, 1, \vee>$ is the semilattice of ***propositional formulae*** with generators $\xi_0, \xi_1, \xi_2, \ldots, \xi_n, \ldots$.

(Step 2) $\mathbb{S}$ is the family of all ***subsemilattices*** of P ; thus

$$\mathbb{S} = \{A \subseteq P \mid 0,1 \in A \;\&\; \forall a,b \in A [a \vee b \in A]\}.$$

Note: $\mathbb{S}$ is an algebraic lattice with a countable number of finite elements. (Why?)

(Step 3) For $A \in \mathbb{S}$, let $\|A\| = \{\downarrow(X \cap A) \mid X \in \|\mathcal{P}\|\}$.

Note: $\|A\|$ is a subdomain of $\|P\| = \|\mathcal{P}\|$, and every countably based domain is isomorphic one such. The semilattice structure of $\|P\|$ is defined by $X \vee Y = \{x \vee y \mid x \in X \;\&\; y \in Y\}$.

Gödel Numbering & Pairing

Theorem. There is a numbering of the elements of P so that all Boolean operations are ***primitive recursive***.

Theorem. Under this numbering, there is a primitive recursive ***pairing operation*** $\langle\!\langle p,q \rangle\!\rangle$ on P with a recursive range where:

(i) $\langle\!\langle 0,0 \rangle\!\rangle = 0$;

(ii) $\langle\!\langle p,q \rangle\!\rangle = 1 \Leftrightarrow p = 1$ or $q = 1$;

(iii) $\langle\!\langle p_0,q_0 \rangle\!\rangle \vee \langle\!\langle p_1,q_1 \rangle\!\rangle = \langle\!\langle p_0 \vee p_1, q_0 \vee q_1 \rangle\!\rangle$;

(iv) $\langle\!\langle p_0,q_0 \rangle\!\rangle \leq \langle\!\langle p_1,q_1 \rangle\!\rangle \Leftrightarrow p_1 = 1$ or $q_1 = 1$ or $[p_0 \leq p_1$ & $q_0 \leq q_1]$.

Hint: Define Boolean injections $\sigma_0, \sigma_1 : P \rightarrowtail P$ **by** $\sigma_0(\xi_n) = \xi_{2n}$ **and** $\sigma_1(\xi_n) = \xi_{2n+1}$. **Then define** $\langle\!\langle p,q \rangle\!\rangle = \sigma_0(p) \vee \sigma_1(q)$.

Another Construction of $\mathcal{P}$

(Step 1) For any set S, let $F(S)$ denote the collection of all the ***finite subsets*** of S.

(Step 2) $F(S)$ may be regarded as a ***vector space*** over the field {0,1}, where the ***zero vector***, 0, is the empty set, and where ***vector addition***, +, is the symmetric difference of sets.
A ***basis*** for the space $F(S)$ consists of the singleton subsets.

(Step 3) Let $P = F(F(\mathbb{N}))$, and define a ***bilinear multiplication*** on P by the stipulation $\{s\}\cdot\{t\} = \{s \cup t\}$ for $s,t \in F(\mathbb{N})$. Let $1 = \{0\}$.

Theorem: The algebra $\langle P, 0, 1, +, \cdot \rangle$ is the ***free Boolean ring*** (with unit) on the generators $\{\{n\}\}$ for $n \in \mathbb{N}$. It can be made into a semilattice by defining $x \vee y = x + y + x\cdot y$.

Note: **Using $P = F(F(\mathbb{N}))$ gives us another Gödel numbering.**

Computable Domains and Mappings

Definition. The ***computable*** elements of $\mathbb{S}$ are those which are ***recursively enumerable*** subsets of P.

Definition. The ***computable*** elements of $||A||$ are those which are ***recursively enumerable*** subsets of P.

Definition. The ***computable*** mappings $F{:}||A||\rightarrow||B||$ are those which are continuous and where the relationship $\downarrow b \subseteq F(\downarrow a)$ between finite elements of $||A||$ and $||B||$ is ***recursively enumerable***.

Domain Products

Definitions. (i) $X \times Y = \{\langle\!\langle p,q \rangle\!\rangle \mid p \in X \ \&\ q \in Y\}$;

(ii) $H = \{\langle\!\langle p,q \rangle\!\rangle \mid [p = 0 \ \&\ q = 0] \text{ or } [p \neq 0 \ \&\ q \neq 0]\}$;

(iii) $A \times_s B = (A \times B) \cap H$.

Lemma. (i) If $A, B \in \mathbb{S}$, then $H, (A \times B), (A \times_s B) \in \mathbb{S}$.

(ii) If $X, Y \in \|P\|$, then $(X, Y) = \downarrow(X \times Y) \in \|P\|$.

Theorem. If $A, B \in \mathbb{S}$, then $\|A \times B\|$ is isomorphic to the ***product*** of the domains $\|A\|$ and $\|B\|$, while $\|A \times_s B\|$ is isomorphic to the ***smash product***.

Hint: Let $X = \{p \mid \langle\!\langle p,0 \rangle\!\rangle \in Z\}$ and $Y = \{q \mid \langle\!\langle 0,q \rangle\!\rangle \in Z\}$, for any $Z \in \|A \times B\|$. Then $X \in \|A\|, Y \in \|B\|$, and $\downarrow(X \times Y) = Z$.

Domain Sums

Definition. Let $\zeta_n = \neg\xi_0 \vee \neg\xi_1 \vee \neg\xi_2 \vee \ldots \vee \neg\xi_{n-1} \vee \xi_n$.

Definition.

$$A_0 + A_1 + A_2 + \ldots + A_n = \{0\} \cup \bigcup_{i \leq n} \{ \langle\!\langle p, \zeta_i \rangle\!\rangle \mid p \in A_i \}.$$

Definition.

$$A_0 +_c A_1 +_c A_2 +_c \ldots +_c A_n = (A_0 + A_1 + A_2 + \ldots + A_n) \cap H.$$

Theorem. For $A_0, A_1, A_2, \ldots, A_n \in \mathbb{S}$, $\|A_0 + A_1 + A_2 + \ldots + A_n\|$ is isomorphic to the ***separated sum*** of the domains $\|A_i\|$.

Theorem. For $A_0, A_1, A_2, \ldots, A_n \in \mathbb{S}$, $\|A_0 +_c A_1 +_c A_2 +_c \ldots +_c A_n\|$ is isomorphic to the ***coalesced sum*** of the domains $\|A_i\|$.

Lifting and Dropping

Definitions. (i) $A_\perp = \{0\} \cup \{ \langle\!\langle p, 0 \rangle\!\rangle \vee \xi_1 \mid p \in A \}$.

(ii) $A^\top = \{1\} \cup \{ \langle\!\langle p, 0 \rangle\!\rangle \wedge \xi_1 \mid p \in A \}$.

Theorem. For $A \in \mathbb{S}$, we have $A_\perp, A^\top \in \mathbb{S}$, and the domain $\|A_\perp\|$ is like $\|A\|$ but with a ***new bottom element***, and $\|A^\top\|$ is like $\|A\|$ but with a ***new top element***.

Note: **All the operations of products, sums, lifts and drops on $\mathbb{S}$ need to be checked for continuity and computability.**

Function Spaces

Theorem. Under the numbering of P, there is a primitive recursive operation $(p \Rightarrow q)$ on P, defined when $p \neq 1$, such that:

(i) $(p \Rightarrow 1) = 1$;

(ii) $\bigvee_{i<k} (p_i \Rightarrow q_i) = 1 \;\Rightarrow\; \exists r \neq 1.\ \bigvee\{q_i \mid p_i \leq r\} = 1$; and

(iii) $(r \Rightarrow s) \leq \bigvee_{i<k} (p_i \Rightarrow q_i) \Leftrightarrow \bigvee_{i<k} (p_i \Rightarrow q_i) = 1$ or $s \leq \bigvee\{q_i \mid p_i \leq r\}$.

Definitions.

(i) $(A \Rightarrow B) = \{\bigvee_{i<k}(p_i \Rightarrow q_i) \mid \forall i < k\,[p_i \in A \backslash \{1\}\ \&\ q_i \in B]\}$;

(ii) $(A \Rightarrow_S B) = \{0,1\} \cup \{\bigvee_{i<k}(p_i \Rightarrow q_i) \mid \forall i < k\,[p_i \in A \backslash \{0,1\}\ \&\ q_i \in B \backslash \{0,1\}]\}$;

Theorem. If $A, B \in \mathbb{S}$, then $(A \Rightarrow B), (A \Rightarrow_S B) \in \mathbb{S}$, and $||A \Rightarrow B||$ is isomorphic to the domain of ***continuous functions*** from $||A||$ to $||B||$ and $||A \Rightarrow_S B||$ gives ***strict continuous functions***.

Embedding $\mathbb{S}$ into $\|P\|$

Note: The finite elements of $\mathbb{S}$ are the finite subsemilattices of P, and P is the (non-finite) unit element of $\mathbb{S}$. The semilattice operation on $\mathbb{S}$ is $A \vee B$.

Lemma. For finite $A, B \in \mathbb{S}$, we have

(i) $\bigvee_{p \in A \setminus \{1\}} (p \Rightarrow p) = \bigvee_{q \in B \setminus \{1\}} (q \Rightarrow q) \Leftrightarrow A = B$; and

(ii) $\bigvee_{r \in (A \vee B) \setminus \{1\}} (r \Rightarrow r) = \bigvee_{p \in A \setminus \{1\}} (p \Rightarrow p) \vee \bigvee_{q \in B \setminus \{1\}} (q \Rightarrow q)$.

Theorem. The domain $\mathbb{S}$ is isomorphic to a subdomain of $\|P \Rightarrow_S P\|$ by a computable embedding.

Recursive Domain Equations

Theorem. All the operations $(A \times B)$, $(A \times_s B)$, $(A + B)$, $(A +_c B)$, $A_\perp$, $A^\top$, $(A \Rightarrow B)$, and $(A \Rightarrow_s B)$ map $\mathbb{S}$ to $\mathbb{S}$ and are ***continuous and computable***. Hence, so are any ***compositions*** of these mappings.

Note: **The standard Fixed-Point Theorem can then be applied to obtain recursively defined domains.**

Definition. $\mathbb{N}_\perp = \{0,1\} \cup \{\zeta_n \mid n \in \mathbb{N}\}$.

Theorem. $\mathbb{N}_\perp \in \mathbb{S}$, and $\| \mathbb{N}_\perp \|$ is isomorphic to the ***flat domain of integers***.

Note: **A typical example of a recursively defined domain is given by $D = \mathbb{N}_\perp +_c (D \Rightarrow D)_\perp$. This constructs a model of the λ-calculus closely related to recursive function theory.**

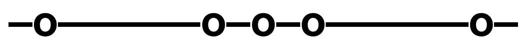
Some Additional Examples

Potentially infinite:

$B = B + B$ binary sequences

$S = \mathbb{N}_\bot \times_s S_\bot$ sequences of integers

$L = \mathbb{N}_\bot \times_s (L \times L)_\bot$ labelled trees



Another lambda-calculus domain:

$A =$ given

$B = B \times (A \times B)$ fixed point

$C = A \times B \cong (A \times B) \times (A \times B) = C \times C$ defined & isomorphed

$D = D \Rightarrow C$ fixed point

$D \times D = (D \Rightarrow C) \times (D \Rightarrow C) \cong D \Rightarrow (C \times C) \cong D \Rightarrow C = D$ isomorphism

$D \Rightarrow D = D \Rightarrow (D \Rightarrow C) \cong (D \times D) \Rightarrow C \cong D \Rightarrow C = D$ isomorphism

$\|P\|$ as a λ-Calculus Model

Definitions.

(i) $\mathbf{Id}(Z) = \downarrow\{\bigvee_{i<k} q_i \mid \forall i<k[q_i \in Z]\}$; and

(ii) $F(X) = \mathbf{Id}(\{q \mid (p \Rightarrow q) \in F \;\&\; p \in X\})$; and

(iii) $\lambda X.\Phi(X) = \mathbf{Id}(\{(p \Rightarrow q) \mid q \in \Phi(\downarrow p) \;\&\; p \neq 1\})$, where

$F, X \in \|P\|$ and $\Phi : \|P\| \rightarrow \|P\|$ is continuous.

Note: **It is possible that $\mathbf{Id}(Z) = P$, if Z is inconsistent.**

However, for F, X, Φ as above, both

$F(X)$ and $\lambda X.\Phi(X)$ are consistent.

Equilogical Spaces

Remember: • The space $||P||$ is not only universal for domains, but it contains as subspaces all countably based T_0-spaces.

• Moreover, by passing to partial equivalence relations (PERs) and equivalence-preserving continuous mappings, we obtain a cartesian closed category (and more).

• It contains the two previous categories and has an intrinsic notion of computable function and computable element. (But a certain subcategory may be better.)

• In this way we have a semantics for a notion of computability at higher types.

Martin-Löf Type Theory

Definition. A ***family of types*** consists of a PER $\mathcal{A}$ and a mapping $\mathcal{B} : ||P|| \to$ PER, where for all $X_0, X_1 \in ||P||$ we have

$$X_0 \; \mathcal{A} \; X_1 \Rightarrow \mathcal{B}(X_0) = \mathcal{B}(X_1).$$

Definition. A ***product*** of a family of types is the PER defined by:

$$F_0 \, (\prod X{:}\mathcal{A}.\, \mathcal{B}(X)) \; F_1 \Leftrightarrow \forall \, X_0, X_1 [X_0 \; \mathcal{A} \; X_1 \Rightarrow F_0(X_0) \; \mathcal{B}(X_0) \; F_1(X_1) \,].$$

Definition. A ***sum*** of a family of types is the PER defined by:

$$Z_0 \, (\sum X{:}\mathcal{A}.\, \mathcal{B}(X)) \; Z_1 \Leftrightarrow \exists \; X_0, X_1, Y_0, Y_1 [\, X_0 \; \mathcal{A} \; X_1 \;\&\; Y_0 \; \mathcal{B}(X_0) \; Y_1 \;\&$$

$$Z_0 = (X_0, Y_0) \;\&\; Z_1 = (X_1, Y_1) \,].$$

Note: It must be proved that these definitions actually do produce PERs.

Systems of Dependent Types

Theorem. Given families of types where always

$$X_0 \mathcal{A} X_1 \Rightarrow \mathcal{B}(X_0) = \mathcal{B}(X_1) \text{ and}$$

$$X_0 \mathcal{A} X_1 \;\&\; Y_0 \mathcal{B}(X_0) Y_1 \Rightarrow \mathcal{C}(X_0,Y_0) = \mathcal{C}(X_1,Y_1) \text{ and}$$

$$X_0 \mathcal{A} X_1 \;\&\; Y_0 \mathcal{B}(X_0) Y_1 \;\&\; Z_0 \mathcal{C}(X_0,Y_0) Z_1 \Rightarrow \mathcal{D}(X_0,Y_0,Z_0) = \mathcal{D}(X_1,Y_1,Z_1),$$

then this ***iterated*** products of sums of ... is a PER:

$$\prod X{:}\mathcal{A}. \sum Y{:}\ \mathcal{B}(X). \prod Z{:}\ \mathcal{C}(X,Y).\ \mathcal{D}(X,Y,Z).$$

Note: Of course, properties of sums and products have to be established showing they follow the usual rules of type theory.

Extensional Identity Types

Definition. Given a PER $\mathcal{A}$ we define:

$$U \ \mathrm{Id}_{\mathcal{A}}(X,Y) \ V \Leftrightarrow U \ \mathcal{A} \ X \ \mathcal{A} \ Y \ \mathcal{A} \ V \ .$$

Proposition. Given a PER $\mathcal{A}$, then $\mathrm{Id}_{\mathcal{A}}(X,Y)$ is a PER and

$$X_0 \ \mathcal{A} \ X_1 \ \& \ Y_0 \ \mathcal{A} \ Y_1 \Rightarrow \mathrm{Id}_{\mathcal{A}}(X_0,Y_0) = \mathrm{Id}_{\mathcal{A}}(X_1,Y_1).$$

Example: **In case** $F \ (\prod X{:}\mathcal{A}. \prod Y{:}\mathcal{A}. \ \mathcal{A}) \ F$**, we can regard** F **as a binary operation of type** $\mathcal{A}$**. Then, if the following type is inhabited, we can say** F **is an associative operation:**

$$\prod X{:}\mathcal{A}. \prod Y{:}\mathcal{A}. \prod Z{:}\mathcal{A}. \ \mathrm{Id}_{\mathcal{A}}(\ F(F(X)(Y))(Z), \ F(X)(F(Y)(Z)) \ ).$$

Why Domain Theory?

The aim of Domain Theory is:

(a) To provide one convenient category having many familiar examples;

(b) To permit some new space constructions, including function spaces;

(c) To allow for the solution of recursive domain equations, including for the λ-calculus; and

(d) To give one sound basis for some notions of higher-type computability.

Why Semilattices?

Semilattice theory:

(a) Has a very elementary definition;

(b) Has a universal space with very easy computable structure;

(c) Has a very direct way to pass to completions; and

(d) Has a category construction needing a minimum of set theory and abstract algebra.

THE END

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