Deciding Equality in Free Categories

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Prove that if G is a graph where equality of edges is decidable, then equality of arrows in the free category $F(G)$ generated by G is also decidable.

More constructively stated, given a function deciding equality of edges in G, construct an algorithm that decides when two morphisms in $F(G)$ that have the same domain and codomain are equal.

For completeness, recall that the free category $F(G)$ is defined to have as objects the vertices of G and morphism inductively generated by the following rules:

$$\frac{e \in G(A,B)}{e \in F(G)(A,B)} \qquad \frac{A \in G_0}{id_A \in F(G)(A,A)} \qquad \frac{f \in F(G)(A,B) \qquad g \in F(G)(B,C)}{g \circ f \in F(G)(A,C)}$$

Quotiented by the equivalence relation generated by the following rules (domains and codomains are elided but the rules only apply when the two sides of an equation have the same domain and codomain):

$$f = f \qquad \frac{f = g \qquad g = h}{f = h} \qquad \frac{f = g}{g = f} \qquad \frac{f = f' \qquad g = g'}{g \circ f = g' \circ f'}$$

$$h \circ (g \circ f) = (h \circ g) \circ f \qquad f \circ \mathrm{id}_A = f \qquad \mathrm{id}_B \circ f = f$$

With identity defined by the syntactic identity id_A and composition defined by the syntactic composition $\circ$. If you prefer to think in set theory, the arrows of this category are technically *equivalence classes* of syntactic arrows, in which case you the exercise is to construct an algorithm that decides when two syntactic arrows are related by the equality relation generated by the rules above, and therefore in the same equivalence class in the free category.